

COLOMBEAU GENERALIZED GEVREY ULTRADISTRIBUTIONS AND THEIR MICROLOCAL ANALYSIS

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ABSTRACT. The purpose of this paper is to construct and to study algebras of generalized Gevrey ultradistributions. We define the generalized Gevrey wave front and give its main properties. As a fundamental application, the well known Hörmander's theorem on the product of two distributions is extended to the case of generalized Gevrey ultradistributions

1. INTRODUCTION

The nonlinear theory of generalized functions initiated by J. F. Colombeau, [4] and [5], in connection with the problem of multiplication of Schwartz distributions [25], has been developed and applied in nonlinear and linear problems, [3], [6], [21] and [20]. The recent book [10] gives further developments and applications of Colombeau generalized functions. Some methods of constructing algebras of generalized functions of Colombeau type are given in [1] and [19]. The proceedings [7] and [13] present different results on nonlinear analysis of Colombeau generalized functions.

Ultradistributions, important in theoretical as well applied fields, see [17], [18] and [24], are natural generalization of Schwartz distributions, so it is natural to search for algebras of generalized functions containing ultradistributions, to study and to apply them. This is the purpose of this paper.

We first introduce algebras of generalized Gevrey ultradistributions, such a question is considered in the only papers [12], [23] and [8]. We then develop a Gevrey microlocal analysis suitable for these algebras in the spirit of [14], [24] and [20]. Finally, we give an application through a generalization of Hörmander's theorem on the wave front of the product of two distributions, this is also an extension of the result of [15].

The algebras $\mathcal{G}^s(\Omega)$ of generalized Gevrey ultradistributions are represented by nets of smooth functions f_ε with exponential growth in ε depending on the Gevrey order s , more precisely

$$\mathcal{G}^s(\Omega) = \frac{\mathcal{E}_m^s(\Omega)}{\mathcal{N}^s(\Omega)},$$

where $\mathcal{E}_m^s(\Omega)$ is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$ satisfying for every compact subset K of Ω , $\forall \alpha \in \mathbb{Z}_+^n, \exists C > 0, \exists k > 0, \exists \varepsilon_0 \in]0, 1[$,

$$(1) \quad |\partial^\alpha f_\varepsilon(x)| \leq C \exp\left(k\varepsilon^{-\frac{1}{2s-1}}\right), \forall x \in K, \forall \varepsilon \leq \varepsilon_0,$$

and $\mathcal{N}^s(\Omega)$ is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$ satisfying for every compact K of $\Omega, \forall \alpha \in \mathbb{Z}_+^n, \exists C > 0, \forall k > 0, \exists \varepsilon_0 \in]0, 1[$,

$$(2) \quad |\partial^\alpha f_\varepsilon(x)| \leq C \exp\left(-k\varepsilon^{-\frac{1}{2s-1}}\right), \forall x \in K, \forall \varepsilon \leq \varepsilon_0$$

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We show that $\mathcal{G}^s(\Omega)$ contains the space of Gevrey ultradistributions of order $(3s - 1)$, and the following diagram of embeddings is commutative

$$(3) \quad \begin{array}{ccc} D^s(\Omega) & \rightarrow & \mathcal{G}^s(\Omega) \\ & \searrow & \uparrow \\ & & E'_{3s-1}(\Omega) \end{array}$$

The Gevrey microlocal analysis in the framework of the algebra $\mathcal{G}^s(\Omega)$ consists first in introducing the algebra of regular generalized Gevrey ultradistributions $\mathcal{G}^{s,\infty}(\Omega)$ and the proof of the following fundamental result

$$(4) \quad \mathcal{G}^{s,\infty}(\Omega) \cap E'_{3s-1}(\Omega) = D^s(\Omega)$$

Then, we define the generalized Gevrey wave front of $f \in \mathcal{G}^s(\Omega)$, denoted $WF_g^s(f)$, and give its main properties.

Finally, we give an application of this generalized Gevrey microlocal analysis. The product of two generalized Gevrey ultradistributions always exists, but there is no final description of the generalized wave front of this product. This problem is also still posed in the Colombeau algebra of generalized functions. In [15], the well-known Hörmander's result on the wave front of the product of two distributions, has been extended to the case of two Colombeau generalized functions. We show this result in the case of two generalized Gevrey ultradistributions, namely we obtain the following theorem.

Theorem 1. *Let $f, g \in \mathcal{G}^s(\Omega)$, satisfying $\forall x \in \Omega$,*

$$(x, 0) \notin WF_g^s(f) + WF_g^s(g),$$

then

$$WF_g^s(fg) \subseteq (WF_g^s(f) + WF_g^s(g)) \cup WF_g^s(f) \cup WF_g^s(g)$$

2. GENERALIZED GEVREY ULTRADISTRIBUTIONS

According to the construction of Colombeau algebras of generalized functions, we introduce an algebra of moderate elements and its ideal of null elements depending on the Gevrey order $s > 1$.

Definition 1. *The space of moderate elements, denoted $\mathcal{E}_m^s(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$ satisfying for every compact subset K of Ω , $\forall \alpha \in \mathbb{Z}_+^n, \exists C > 0, \exists k > 0, \exists \varepsilon_0 \in]0, 1[$, such that*

$$(5) \quad |\partial^\alpha f_\varepsilon(x)| \leq C \exp\left(k\varepsilon^{-\frac{1}{2s-1}}\right), \forall x \in K, \forall \varepsilon \leq \varepsilon_0$$

The space of null elements, denoted $\mathcal{N}^s(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{[0,1]}$ satisfying for every compact K of $\Omega, \forall \alpha \in \mathbb{Z}_+^n, \exists C > 0, \forall k > 0, \exists \varepsilon_0 \in]0, 1[$, such that

$$(6) \quad |\partial^\alpha f_\varepsilon(x)| \leq C \exp\left(-k\varepsilon^{-\frac{1}{2s-1}}\right), \forall x \in K, \forall \varepsilon \leq \varepsilon_0$$

The main properties of the spaces $\mathcal{E}_m^s(\Omega)$ and $\mathcal{N}^s(\Omega)$ are given in the following proposition.

Proposition 2. *1) The space of moderate elements $\mathcal{E}_m^s(\Omega)$ is an algebra stable by derivation.*

2) The space $\mathcal{N}^s(\Omega)$ is an ideal of $\mathcal{E}_m^s(\Omega)$.

Proof. 1) Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^s(\Omega)$ and K be a compact of Ω , then $\forall \beta \in \mathbb{Z}_+^n, \exists C_1 = C_1(\beta) > 0, \exists k_1 = k_1(\beta) > 0, \exists \varepsilon_{1\beta} \in]0, 1[$ such that $\forall x \in K, \forall \varepsilon \leq \varepsilon_{1\beta}$,

$$(7) \quad |\partial^\beta f_\varepsilon(x)| \leq C_1 \exp\left(k_1\varepsilon^{-\frac{1}{2s-1}}\right)$$

$\forall \beta \in \mathbb{Z}_+^n, \exists C_2 = C_2(\beta) > 0, \exists k_2 = k_2(\beta) > 0, \exists \varepsilon_{2\beta} \in]0, 1[$, such that $\forall x \in K, \forall \varepsilon \leq \varepsilon_{2\beta}$,

$$(8) \quad |\partial^\beta g_\varepsilon(x)| \leq C_2 \exp\left(k_2\varepsilon^{-\frac{1}{2s-1}}\right)$$

Let $\alpha \in \mathbb{Z}_+^n$, then

$$|\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| \leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta} f_\varepsilon(x)| |\partial^\beta g_\varepsilon(x)|$$

For $k = \max \{k_1(\beta) : \beta \leq \alpha\} + \max \{k_2(\beta) : \beta \leq \alpha\}$, $\varepsilon \leq \min \{\varepsilon_{1\beta}, \varepsilon_{2\beta}; |\beta| \leq |\alpha|\}$ and $x \in K$, we have

$$\begin{aligned} \exp\left(-k\varepsilon^{-\frac{1}{2s-1}}\right) |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \exp\left(-k_1\varepsilon^{-\frac{1}{2s-1}}\right) |\partial^{\alpha-\beta} f_\varepsilon(x)| \times \\ &\quad \times \exp\left(-k_2\varepsilon^{-\frac{1}{2s-1}}\right) |\partial^\beta g_\varepsilon(x)| \\ &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} C_1(\alpha - \beta) C_2(\beta) = C(\alpha), \end{aligned}$$

i.e. $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{E}_m^s(\Omega)$.

It is clear, from (7) that for every compact K of Ω , $\forall \beta \in \mathbb{Z}_+^n$, $\exists C_1 = C_1(\beta + 1) > 0$, $\exists k_1 = k_1(\beta + 1) > 0$, $\exists \varepsilon_{1\beta} \in]0, 1[$ such that $\forall x \in K, \forall \varepsilon \leq \varepsilon_{1\beta}$,

$$|\partial^\beta (\partial f_\varepsilon)(x)| \leq C_1 \exp\left(k_1\varepsilon^{-\frac{1}{2s-1}}\right),$$

i.e. $(\partial f_\varepsilon)_\varepsilon \in \mathcal{E}_m^s(\Omega)$.

2) If $(g_\varepsilon)_\varepsilon \in \mathcal{N}^s(\Omega)$, for every K compact of Ω , $\forall \beta \in \mathbb{Z}_+^m$, $\exists C_2 = C_2(\beta) > 0$, $\forall k_2 > 0$, $\exists \varepsilon_{2\beta} \in]0, 1[$,

$$(9) \quad |\partial^\alpha g_\varepsilon(x)| \leq C_2 \exp\left(-k_2\varepsilon^{-\frac{1}{2s-1}}\right), \forall x \in K, \forall \varepsilon \leq \varepsilon_{2\beta}$$

Let $\alpha \in \mathbb{Z}_+^m$ and $k > 0$, then

$$\begin{aligned} \exp\left(k\varepsilon^{-\frac{1}{2s-1}}\right) |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \exp\left(k\varepsilon^{-\frac{1}{2s-1}}\right) \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta} f_\varepsilon(x)| \times \\ &\quad \times |\partial^\beta g_\varepsilon(x)| \end{aligned}$$

Let $k_2 = \max \{k_1(\alpha - \beta) : \beta \leq \alpha\} + k$ and $\varepsilon \leq \min \{\varepsilon_{1\beta}, \varepsilon_{2\beta}; \beta \leq \alpha\}$, then $\forall x \in K$,

$$\begin{aligned} \exp\left(k\varepsilon^{-\frac{1}{2s-1}}\right) |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left[\exp\left(-k_1\varepsilon^{-\frac{1}{2s-1}}\right) |\partial^{\alpha-\beta} f_\varepsilon(x)| \right. \\ &\quad \left. \exp\left(k_2\varepsilon^{-\frac{1}{2s-1}}\right) |\partial^\beta g_\varepsilon(x)| \right] \\ &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} C_1(\alpha - \beta) C_2(\beta) = C(\alpha), \end{aligned}$$

which shows that $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{N}^s(\Omega)$ □

Remark 1. The algebra of moderate elements $\mathcal{E}_m^s(\Omega)$ is not necessary stable by s -ultradifferentiable operators.

Definition 2. The algebra of generalized Gevrey ultradistributions of order $s > 1$, denoted $\mathcal{G}^s(\Omega)$, is the quotient algebra

$$(10) \quad \mathcal{G}^s(\Omega) = \frac{\mathcal{E}_m^s(\Omega)}{\mathcal{N}^s(\Omega)}$$

Remark 2. We have $\mathcal{N}^s(\Omega) \subset \mathcal{N}(\Omega) \subset \mathcal{E}_m(\Omega) \subset \mathcal{E}_m^s(\Omega)$, where $\mathcal{N}(\Omega)$ is the Colombeau algebra of null elements and $\mathcal{E}_m(\Omega)$ the Colombeau algebra of moderate elements.

Our construction permits also to define the Gevrey type algebra of generalized complex numbers, denoted \mathcal{C}^s , by

$$\mathcal{C}^s = \frac{\mathcal{E}_0^s}{\mathcal{N}_0^s},$$

where

$$\mathcal{E}_0^s = \left\{ (a_\varepsilon)_\varepsilon \in \mathbb{C}^{[0,1[}; \exists k > 0, |a_\varepsilon| = O\left(\exp\left(k\varepsilon^{-\frac{1}{2s-1}}\right)\right) \right\}$$

and

$$\mathcal{N}_0^s = \left\{ (a_\varepsilon)_\varepsilon \in \mathbb{C}^{[0,1[}; \forall k > 0, |a_\varepsilon| = O\left(\exp\left(-k\varepsilon^{-\frac{1}{2s-1}}\right)\right) \right\}$$

Some local properties of the algebra $\mathcal{G}^s(\Omega)$ can be studied thanks to the algebra of generalized complex numbers \mathcal{C}^s .

3. EMBEDDING OF GEVREY ULTRADISTRIBUTIONS

We recall some definitions and results on Gevrey ultradistributions, see [17], [18] or [24].

Definition 3. A function $f \in E^s(\Omega)$ if $f \in C^\infty(\Omega)$ and for every compact subset K of Ω , $\exists c > 0, \forall \alpha \in \mathbb{Z}_+^m$,

$$(11) \quad \sup_{x \in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1} (\alpha!)^s$$

Obviously we have $E^t(\Omega) \subset E^s(\Omega)$ if $1 \leq t \leq s$. It is well known that $E^1(\Omega) = A(\Omega)$ is the space of all real analytic functions in Ω and if we denote by $D^s(\Omega)$ the space $E^s(\Omega) \cap C_0^\infty(\Omega)$, then $D^s(\Omega)$ is non trivial if and only if $s > 1$. The topological dual of $D^s(\Omega)$, denoted $D'_s(\Omega)$, is called the space of Gevrey ultradistributions of order s . The space $E'_s(\Omega)$ is the topological dual of $E^s(\Omega)$ and is identified with the space of Gevrey ultradistributions with compact supports.

Definition 4. A differential operator of infinite order $P(D) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma$ is called an ultradifferential operator of class s or s -ultradifferential operator, if for every $h > 0$ there exists $c > 0$ such that $\forall \gamma \in \mathbb{Z}_+^m$,

$$(12) \quad |a_\gamma| \leq c \frac{h^{|\gamma|}}{(\gamma!)^s}$$

The importance of s -ultradifferential operator is in the following result.

Proposition 3. Let $T \in E'_s(\Omega)$ and $\text{supp} T \subset K$, then there exists an s -ultradifferential operator $P(D) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma$, $M > 0$ and continuous functions $f_\gamma \in C_0(K)$ such that $\sup_{\gamma \in \mathbb{Z}_+^m, x \in \mathbb{R}^m} |f_\gamma(x)| \leq M$ and

$$(13) \quad T = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma f_\gamma$$

The space $\mathcal{S}^{(s)}$, $s > 1$, see [11], is the space of functions $\varphi \in C^\infty(\Omega)$ such that $\forall b > 0$, we have

$$(14) \quad \sigma_{b,s}(\varphi) = \sup_{\alpha, \beta \in \mathbb{Z}_+^m} \int \frac{|x|^{|\beta|}}{b^{|\alpha+\beta|} \alpha!^s \beta!^s} |\partial^\alpha \varphi(x)| dx < \infty$$

Lemma 4. There exists a $\phi \in \mathcal{S}^{(s)}$ satisfying

$$\int \phi(x) dx = 1 \text{ and } \int x^\alpha \phi(x) dx = 0, \forall \alpha \in \mathbb{Z}_+^m \setminus \{0\}$$

Proof. For an example of function $\phi \in \mathcal{S}^{(s)}$ satisfying these conditions, take the Fourier transform of a function of the class $D^{(s)}(\Omega)$ equals 1 in neighborhood of the origin. Here $D^{(s)}(\Omega)$ denotes the projective Gevrey space of order s , i. e. $D^{(s)}(\Omega) = E^{(s)}(\Omega) \cap C_0^\infty(\Omega)$, where $f \in E^{(s)}(\Omega)$, if $f \in C^\infty(\Omega)$ and for every compact subset K of $\Omega, \forall b > 0, \exists c > 0, \forall \alpha \in \mathbb{Z}_+^m$,

$$(15) \quad \sup_{x \in K} |\partial^\alpha f(x)| \leq cb^{|\alpha|} (\alpha!)^s$$

□

Definition 5. The net $\phi_\varepsilon = \varepsilon^{-m} \phi(\cdot/\varepsilon), \varepsilon \in]0, 1[$, where ϕ satisfies the Lemma, is called a net of mollifiers.

The space $E^s(\Omega)$ is embedded into $\mathcal{G}^s(\Omega)$ by the standard canonical injection

$$I : \begin{array}{ccc} E^s(\Omega) & \rightarrow & \mathcal{G}^s(\Omega) \\ f & \rightarrow & [f] = cl(f_\varepsilon), \end{array}$$

where $f_\varepsilon = f, \forall \varepsilon \in]0, 1[$.

The following proposition gives the natural embedding of Gevrey ultradistributions into $\mathcal{G}^s(\Omega)$.

Theorem 5. The map

$$(16) \quad \begin{array}{ccc} J : E'_{3s-1}(\Omega) & \rightarrow & \mathcal{G}^s(\Omega) \\ T & \rightarrow & [T] = cl\left((T * \phi_\varepsilon)_{/\Omega}\right)_\varepsilon \end{array}$$

is an embedding.

Proof. Let $T \in E'_{3s-1}(\Omega)$ with $\text{supp} T \subset K$, then there exists an $(3s-1)$ -ultradifferential operator $P(D) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma$ and continuous functions f_γ with $\text{supp} f_\gamma \subset K, \forall \gamma \in \mathbb{Z}_+^m$, and $\sup_{\gamma \in \mathbb{Z}_+^m, x \in K} |f_\gamma(x)| \leq M$, such that

$$(17) \quad T = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma D^\gamma f_\gamma$$

We have

$$T * \phi_\varepsilon(x) = \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma \frac{(-1)^{|\gamma|}}{\varepsilon^{|\gamma|}} \int f_\gamma(x + \varepsilon y) D^\gamma \phi(y) dy$$

Let $\alpha \in \mathbb{Z}_+^m$, then

$$|\partial^\alpha (T * \phi_\varepsilon(x))| \leq \sum_{\gamma \in \mathbb{Z}_+^m} a_\gamma \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x + \varepsilon y)| |D^{\gamma+\alpha} \phi(y)| dy$$

From (12) and the inequality

$$(18) \quad (\beta + \alpha)!^t \leq 2^{t|\beta+\alpha|} \alpha!^t \beta!^t, \quad \forall t \geq 1,$$

we have, $\forall h > 0, \exists c > 0$, such that

$$\begin{aligned} |\partial^\alpha (T * \phi_\varepsilon(x))| &\leq \sum_{\gamma \in \mathbb{Z}_+^m} c \frac{h^{|\gamma|}}{\gamma!^{3s-1}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int |f_\gamma(x + \varepsilon y)| |D^{\gamma+\alpha} \phi(y)| dy \\ &\leq \sum_{\gamma \in \mathbb{Z}_+^m} c \alpha!^{3s-1} \frac{2^{(3s-1)|\gamma+\alpha|} h^{|\gamma|}}{(\gamma + \alpha)!^{2s-1}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} b^{|\gamma+\alpha|} \times \\ &\quad \times \int |f_\gamma(x + \varepsilon y)| \frac{|D^{\gamma+\alpha} \phi(y)|}{b^{|\gamma+\alpha|} (\gamma + \alpha)!^s} dy, \end{aligned}$$

then for $h > \frac{1}{2}$,

$$\begin{aligned} \frac{1}{\alpha!^{3s-1}} |\partial^\alpha (T * \phi_\varepsilon(x))| &\leq \sigma_{b,s}(\phi) Mc \sum_{\gamma \in \mathbb{Z}_+^m} 2^{-|\gamma|} \frac{(2^{3s}bh)^{|\gamma+\alpha|}}{(\gamma+\alpha)!^{2s-1}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \\ &\leq C \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right), \end{aligned}$$

i.e.

$$(19) \quad |\partial^\alpha (T * \phi_\varepsilon(x))| \leq C(\alpha) \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right),$$

where $k_1 = (2s-1)(2^{3s}bh)^{\frac{1}{2s-1}}$.

Suppose that $(T * \phi_\varepsilon)_\varepsilon \in \mathcal{N}^s(\Omega)$, then for every compact L of Ω , $\exists C > 0$, $\forall k > 0$, $\exists \varepsilon_0 \in]0, 1[$,

$$(20) \quad |T * \phi_\varepsilon(x)| \leq C \exp\left(-k \varepsilon^{-\frac{1}{2s-1}}\right), \forall x \in L, \forall \varepsilon \leq \varepsilon_0$$

Let $\chi \in D^{3s-1}(\Omega)$ and $\chi = 1$ in neighborhood of K , then $\forall \psi \in E^{3s-1}(\Omega)$,

$$\langle T, \psi \rangle = \langle T, \chi \psi \rangle = \lim_{\varepsilon \rightarrow 0} \int (T * \phi_\varepsilon)(x) \chi(x) \psi(x) dx$$

Consequently, from (20), we obtain

$$\left| \int (T * \phi_\varepsilon)(x) \chi(x) \psi(x) dx \right| \leq C \exp\left(-k \varepsilon^{-\frac{1}{2s-1}}\right), \forall \varepsilon \leq \varepsilon_0,$$

which gives $\langle T, \psi \rangle = 0$ □

Remark 3. We have $C(\alpha) = \alpha!^{3s-1} \sigma_{b,s}(\phi) Mc$ in (19).

In order to show that the following diagram of embeddings

$$\begin{array}{ccc} D^s(\Omega) & \rightarrow & \mathcal{G}^s(\Omega) \\ & \searrow & \uparrow \\ & & E'_{3s-1}(\Omega) \end{array}$$

is commutative, we have to prove the following fundamental result.

Proposition 6. Let $f \in D^s(\Omega)$ and $(\phi_\varepsilon)_\varepsilon$ be a net of mollifiers, then

$$\left(f - (f * \phi_\varepsilon)_{/\Omega} \right)_\varepsilon \in \mathcal{N}^s(\Omega)$$

Proof. Let $f \in D^s(\Omega)$, then there exists a constant $C > 0$, such that

$$(21) \quad |\partial^\alpha f(x)| \leq C^{|\alpha|+1} \alpha!^s, \forall \alpha \in \mathbb{Z}_+^m, \forall x \in \Omega$$

Let $\alpha \in \mathbb{Z}_+^m$, the Taylor formula and the properties of ϕ_ε give

$$\partial^\alpha (f * \phi_\varepsilon - f)(x) = \sum_{|\beta|=N} \int \frac{(\varepsilon y)^\beta}{\beta!} \partial^{\alpha+\beta} f(\xi) \phi(y) dy,$$

where $x \leq \xi \leq x + \varepsilon y$. Consequently, for $b > 0$, we have

$$\begin{aligned} |\partial^\alpha (f * \phi_\varepsilon - f)(x)| &\leq \varepsilon^N \sum_{|\beta|=N} \int \frac{|y|^N}{\beta!} |\partial^{\alpha+\beta} f(\xi)| |\phi(y)| dy \\ &\leq \alpha!^s \varepsilon^N \sum_{|\beta|=N} \beta!^{2s-1} 2^{s|\alpha+\beta|} b^{|\beta|} \int \frac{|\partial^{\alpha+\beta} f(\xi)|}{(\alpha+\beta)!^s} \times \\ &\quad \times \frac{|y|^{|\beta|}}{b^{|\beta|} \beta!^s} |\phi(y)| dy \end{aligned}$$

Let $k > 0$ and $T > 0$, then

$$\begin{aligned}
|\partial^\alpha (f * \phi_\varepsilon - f)(x)| &\leq \alpha!^s (\varepsilon N^{2s-1})^N (k^{2s-1}T)^{-N} \times \\
&\quad \times \sum_{|\beta|=N} \int 2^{s|\alpha+\beta|} (k^{2s-1}bT)^{|\beta|} \frac{|\partial^{\alpha+\beta} f(\xi)|}{(\alpha+\beta)!^s} \times \\
&\quad \times \frac{|y|^{|\beta|}}{b^{|\beta|}\beta!^s} |\phi(y)| dy \\
&\leq \alpha!^s (\varepsilon N^{2s-1})^N (k^{2s-1}T)^{-N} \times \\
&\quad \times C\sigma_{b,s}(\phi) (2^s C)^{|\alpha|} \sum_{|\beta|=N} (2^s k^{2s-1}bT)^{|\beta|} C^{|\beta|},
\end{aligned}$$

hence, taking $2^s k^{2s-1}bTC \leq \frac{1}{2a}$, with $a > 1$, we obtain

$$\begin{aligned}
|\partial^\alpha (f * \phi_\varepsilon - f)(x)| &\leq \alpha!^s (\varepsilon N^{2s-1})^N (k^{2s-1}T)^{-N} \times \\
(22) \quad &\quad \times C\sigma_{b,s}(\phi) (2^s C)^{|\alpha|} a^{-N} \sum_{|\beta|=N} \left(\frac{1}{2}\right)^{|\beta|} \\
&\leq \sigma_{b,s}(\phi) C^{|\alpha|+1} \alpha!^s (\varepsilon N^{2s-1})^N (k^{2s-1}T)^{-N} a^{-N}
\end{aligned}$$

Let $\varepsilon_0 \in]0, 1[$ such that $\varepsilon_0^{\frac{1}{2s-1}} \frac{\ln a}{k} < 1$ and take $T > 2^{2s-1}$, then

$$\left(T^{\frac{1}{2s-1}} - 1\right) > 1 > \frac{\ln a}{k} \varepsilon^{\frac{1}{2s-1}}, \forall \varepsilon \leq \varepsilon_0,$$

in particular, we have

$$\left(\frac{\ln a}{k} \varepsilon^{\frac{1}{2s-1}}\right)^{-1} T^{\frac{1}{2s-1}} - \left(\frac{\ln a}{k} \varepsilon^{\frac{1}{2s-1}}\right)^{-1} > 1$$

Then, there exists $N = N(\varepsilon) \in \mathbb{Z}^+$, such that

$$\left(\frac{\ln a}{k} \varepsilon^{\frac{1}{2s-1}}\right)^{-1} < N < \left(\frac{\ln a}{k} \varepsilon^{\frac{1}{2s-1}}\right)^{-1} T^{\frac{1}{2s-1}},$$

i.e.

$$(23) \quad 1 \leq \frac{\ln a}{k} \varepsilon^{\frac{1}{2s-1}} N \leq T^{\frac{1}{2s-1}},$$

which gives

$$a^{-N} \leq \exp\left(-k\varepsilon^{-\frac{1}{2s-1}}\right) \quad \text{and} \quad \frac{\varepsilon N^{2s-1}}{k^{2s-1}T} \leq \left(\frac{1}{\ln a}\right)^{2s-1} < 1,$$

if we choose $\ln a > 1$. Finally, from (22), we have

$$(24) \quad |\partial^\alpha (f * \phi_\varepsilon - f)(x)| \leq C \exp\left(-k\varepsilon^{-\frac{1}{2s-1}}\right),$$

i.e. $f * \phi_\varepsilon - f \in \mathcal{N}^s(\Omega)$ □

From the proof, see (22), we obtained in fact the following result.

Corollary 7. *Let $f \in D^s(\Omega)$, then for every compact K of Ω , $\exists C > 0, \forall \alpha \in \mathbb{Z}_+^m, \forall k > 0, \exists \varepsilon_0 \in]0, 1[, \forall x \in K, \forall \varepsilon \leq \varepsilon_0$,*

$$(25) \quad |\partial^\alpha (f * \phi_\varepsilon - f)(x)| \leq C^{|\alpha|+1} \alpha!^s \exp\left(-k\varepsilon^{-\frac{1}{2s-1}}\right)$$

Let Ω' be an open subset of Ω and let $f = (f_\varepsilon)_\varepsilon + \mathcal{N}^s(\Omega) \in \mathcal{G}^s(\Omega)$, the restriction of f to Ω' , denoted $f|_{\Omega'}$, is defined as

$$(26) \quad (f_{\varepsilon/\Omega'})_\varepsilon + \mathcal{N}^s(\Omega') \in \mathcal{G}^s(\Omega')$$

One can easily prove that the functor $\Omega \rightarrow \mathcal{G}^s(\Omega)$ defines a presheaf in the same way as in the case of the algebra of Colombeau generalized functions $\mathcal{G}(\Omega)$, see for example [10]. Consequently, we introduce the support of $f \in \mathcal{G}^s(\Omega)$, denoted $\text{supp}_g f$, as the complement of the largest open set U such that $f|_U = 0$.

Definition 6. *The space of elements of $\mathcal{G}^s(\Omega)$ with compact supports is denoted $\mathcal{G}_c^s(\Omega)$.*

It is not difficult to prove the following result.

Proposition 8. *1) The space $\mathcal{G}_c^s(\Omega)$ is the space of $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}^s(\Omega)$ satisfying, $\exists K$ a compact subset of Ω , $\exists \varepsilon_0 \in]0, 1[$, $\forall \varepsilon \in]0, \varepsilon_0[$, $\text{supp} f_\varepsilon \subset K$.*

2) Let $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_c^s(\Omega)$, then $\exists C > 0$, $\exists k_1 > 0$, $\exists \varepsilon_0 > 0$, $\forall k_2 > 0$, $\forall \xi \in \mathbb{R}^m$, $\forall \varepsilon \leq \varepsilon_0$,

$$(27) \quad \left| \widehat{f}_\varepsilon(\xi) \right| \leq C \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} + k_2 |\xi|^{\frac{1}{s}} \right)$$

4. REGULAR GENERALIZED GEVREY ULTRADISTRIBUTIONS

To develop a local or microlocal analysis with respect to a "good space of regular elements" one needs first to define these regular elements, then the notion of singular support and its microlocalisation.

Definition 7. *We define $\mathcal{E}_m^{s,\infty}(\Omega)$ the space of regular elements as the space of $(f_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^{]0,1[}$ satisfying, for every compact subset K of Ω , $\exists C > 0$, $\exists k > 0$, $\exists \varepsilon_0 \in]0, 1[$, $\forall \alpha \in \mathbb{Z}_+^n$, $\forall x \in K$, $\forall \varepsilon \leq \varepsilon_0$,*

$$(28) \quad |\partial^\alpha f_\varepsilon(x)| \leq C^{|\alpha|+1} \alpha!^s \exp \left(k \varepsilon^{-\frac{1}{2s-1}} \right)$$

Proposition 9. *1) The space $\mathcal{E}_m^{s,\infty}(\Omega)$ is an algebra stable by the action of s -ultradifferential operators.*

2) The space $\mathcal{N}_^s(\Omega) := \mathcal{N}^s(\Omega) \cap \mathcal{E}_m^{s,\infty}(\Omega)$ is an ideal of $\mathcal{E}_m^{s,\infty}(\Omega)$.*

Proof. 1) Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{s,\infty}(\Omega)$ and K be a compact of Ω , then $\exists c_1 > 0$, $\exists k_1 > 0$, $\exists \varepsilon_1 \in]0, 1[$ such that $\forall x \in K$, $\forall \alpha \in \mathbb{Z}_+^m$, $\forall \varepsilon \leq \varepsilon_1$,

$$(29) \quad |\partial^\alpha f_\varepsilon(x)| \leq c_1^{|\alpha|+1} \alpha!^s \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} \right)$$

We have also $\exists c_2 > 0$, $\exists k_2 > 0$, $\exists \varepsilon_2 \in]0, 1[$ such that $\forall x \in K$, $\forall \alpha \in \mathbb{Z}_+^m$, $\forall \varepsilon \leq \varepsilon_2$,

$$(30) \quad |\partial^\alpha g_\varepsilon(x)| \leq c_2^{|\alpha|+1} \alpha!^s \exp \left(k_2 \varepsilon^{-\frac{1}{2s-1}} \right)$$

Let $\alpha \in \mathbb{Z}_+^n$, then

$$\frac{1}{\alpha!^s} |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| \leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \frac{1}{(\alpha-\beta)!^s} |\partial^{\alpha-\beta} f_\varepsilon(x)| \frac{1}{\beta!^s} |\partial^\beta g_\varepsilon(x)|$$

Let $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$ and $k = k_1 + k_2$, then we have $\forall \alpha \in \mathbb{Z}_+^m, \forall x \in K$,

$$\begin{aligned} \exp\left(-k\varepsilon^{-\frac{1}{2s-1}}\right) \frac{1}{\alpha!^s} |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \frac{\exp\left(-k_1\varepsilon^{-\frac{1}{2s-1}}\right)}{(\alpha-\beta)!^s} |\partial^{\alpha-\beta} f_\varepsilon(x)| \times \\ &\quad \times \frac{\exp\left(-k_2\varepsilon^{-\frac{1}{2s-1}}\right)}{\beta!^s} |\partial^\beta g_\varepsilon(x)| \\ &\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} c_1^{|\alpha-\beta|} c_2^{|\beta|} \\ &\leq 2^{|\alpha|} (c_1 + c_2)^{|\alpha|}, \end{aligned}$$

i. e. $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{s,\infty}(\Omega)$.

Let now $P(D) = \sum a_\gamma D^\gamma$ be an s -ultradifferential operator, then $\forall h > 0, \exists b > 0$, such that

$$\begin{aligned} \exp\left(k_1\varepsilon^{-\frac{1}{2s-1}}\right) \frac{1}{\alpha!^s} |\partial^\alpha (P(D) f_\varepsilon(x))| &\leq \exp\left(k_1\varepsilon^{-\frac{1}{2s-1}}\right) \sum_{\gamma \in \mathbb{Z}_+^m} b \frac{h^{|\gamma|}}{\gamma!^s} \frac{1}{\alpha!^s} |\partial^{\alpha+\gamma} f_\varepsilon(x)| \\ &\leq b \exp\left(k_1\varepsilon^{-\frac{1}{2s-1}}\right) \sum_{\gamma \in \mathbb{Z}_+^m} \frac{2^{s|\alpha+\gamma|} h^{|\gamma|}}{(\alpha+\gamma)!^s} |\partial^{\alpha+\gamma} f_\varepsilon(x)| \\ &\leq b \sum_{\gamma \in \mathbb{Z}_+^m} 2^{s|\alpha+\gamma|} h^{|\gamma|} c_1^{|\alpha+\gamma|}, \end{aligned}$$

hence, for $2^s h c_1 \leq \frac{1}{2}$, we have

$$\exp\left(k\varepsilon^{-\frac{1}{2s-1}}\right) \frac{1}{\alpha!^s} |\partial^\alpha (P(D) f_\varepsilon(x))| \leq c' (2^s c_1)^{|\alpha|},$$

which shows that $(P(D) f_\varepsilon)_\varepsilon \in \mathcal{E}_m^{s,\infty}(\Omega)$.

2) The fact that $\mathcal{N}_*^s(\Omega) = \mathcal{N}^s(\Omega) \cap \mathcal{E}_m^{s,\infty}(\Omega) \subset \mathcal{E}_m^s(\Omega)$ and $\mathcal{N}^s(\Omega)$ is an ideal of $\mathcal{E}_m^s(\Omega)$, then $\mathcal{N}_*^s(\Omega)$ is an ideal of $\mathcal{E}_m^{s,\infty}(\Omega)$ \square

Remark 4. If the inclusion $\mathcal{N}^s(\Omega) \subset \mathcal{E}_m^{s,\infty}(\Omega)$ holds, then $\mathcal{N}_*^s(\Omega) = \mathcal{N}^s(\Omega)$.

Now, we define the Gevrey regular elements of $\mathcal{G}^s(\Omega)$.

Definition 8. The algebra of regular generalized Gevrey ultradistributions of order $s > 1$, denoted $\mathcal{G}^{s,\infty}(\Omega)$, is the quotient algebra

$$\mathcal{G}^{s,\infty}(\Omega) = \frac{\mathcal{E}_m^{s,\infty}(\Omega)}{\mathcal{N}_*^s(\Omega)}$$

Remark 5. It is clear that $E^s(\Omega) \hookrightarrow \mathcal{G}^{s,\infty}(\Omega)$.

Definition 9. We define the $\mathcal{G}^{s,\infty}$ -singular support of a generalized Gevrey ultradistribution $f \in \mathcal{G}^s(\Omega)$, denoted $s\text{-singsupp}_g(f)$, as the complement of the largest open set Ω' such that $f \in \mathcal{G}^{s,\infty}(\Omega')$.

Proposition 10. Let $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_c^s(\Omega)$, then f is regular if and only if $\exists k_1 > 0, \exists k_2 > 0, \exists C > 0, \exists \varepsilon_1 > 0, \forall \varepsilon \leq \varepsilon_1$, such that

$$(31) \quad \left| \widehat{f}_\varepsilon(\xi) \right| \leq C \exp\left(k_1\varepsilon^{-\frac{1}{2s-1}} - k_2|\xi|^{\frac{1}{s}}\right), \forall \xi \in \mathbb{R}^m$$

Proof. Suppose that $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_c^s(\Omega) \cap \mathcal{G}^{s,\infty}(\Omega)$, then $\exists C_1 > 0, \exists k_1 > 0, \exists \varepsilon_1 > 0, \forall \alpha \in \mathbb{Z}_+^n, \forall x \in K, \forall \varepsilon \leq \varepsilon_1$, $\text{supp} f_\varepsilon \subset K$, such that

$$(32) \quad |\partial^\alpha f_\varepsilon| \leq C_1^{|\alpha|+1} \alpha!^s \exp\left(k_1\varepsilon^{-\frac{1}{2s-1}}\right)$$

Consequently we have, $\forall \alpha \in \mathbb{Z}_+^m$,

$$|\xi^\alpha| \left| \widehat{f_\varepsilon}(\xi) \right| \leq \left| \int \exp(-ix\xi) \partial^\alpha f_\varepsilon(x) dx \right|,$$

then, $\exists C > 0, \forall \varepsilon \leq \varepsilon_1$,

$$|\xi|^{|\alpha|} \left| \widehat{f_\varepsilon}(\xi) \right| \leq C^{|\alpha|+1} \alpha!^s \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right)$$

For $\alpha \in \mathbb{Z}_+^m, \exists N \in \mathbb{Z}_+$ such that

$$\frac{N}{s} \leq |\alpha| < \frac{N}{s} + 1,$$

so

$$\begin{aligned} |\xi|^{\frac{N}{s}} \left| \widehat{f_\varepsilon}(\xi) \right| &\leq C^{|\alpha|+1} |\alpha|^{|\alpha|s} \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right) \\ &\leq C^{N+1} N^N \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right) \end{aligned}$$

Hence $\exists C > 0, \forall N \in \mathbb{Z}^+$,

$$\left| \widehat{f_\varepsilon}(\xi) \right| \leq C^{N+1} |\xi|^{-\frac{N}{s}} N! \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right),$$

which gives

$$\left| \widehat{f_\varepsilon}(\xi) \right| \exp\left(\frac{1}{2C} |\xi|^{\frac{1}{s}}\right) \leq C \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right) \sum 2^{-N},$$

or

$$\left| \widehat{f_\varepsilon}(\xi) \right| \leq C' \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}} - \frac{1}{2C} |\xi|^{\frac{1}{s}}\right),$$

i.e. we have (31).

Suppose now that (31) is valid, then $\forall \varepsilon \leq \varepsilon_0$,

$$|\partial^\alpha f_\varepsilon(x)| \leq C_1 \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right) \int |\xi^\alpha| \exp\left(-k_2 |\xi|^{\frac{1}{s}}\right) d\xi$$

Due to the inequality $t^N \leq N! \exp(t), \forall t > 0$, then $\exists C_2 = C(k_2)$ such that

$$|\xi^\alpha| \exp\left(-\frac{k_2}{2} |\xi|^{\frac{1}{s}}\right) \leq C_2^{|\alpha|} \alpha!^s,$$

then

$$\begin{aligned} |\partial^\alpha f_\varepsilon(x)| &\leq C_1 \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right) C_2^{|\alpha|} \alpha!^s \int \exp\left(-\frac{k_2}{2} |\xi|^{\frac{1}{s}}\right) d\xi \\ &\leq C^{|\alpha|+1} \alpha!^s \exp\left(k_1 \varepsilon^{-\frac{1}{2s-1}}\right), \end{aligned}$$

where $C = \max\left(C_1 \int \exp\left(-\frac{k_2}{2} |\xi|^{\frac{1}{s}}\right) d\xi, C_2\right)$, i.e. $f \in \mathcal{G}^{s,\infty}(\Omega)$ □

The algebra $\mathcal{G}^{s,\infty}(\Omega)$ plays the same role as the Oberguggenberger subalgebra of regular elements $\mathcal{G}^\infty(\Omega)$ of the Colombeau algebra $\mathcal{G}(\Omega)$, see [21].

Theorem 11. *We have*

$$\mathcal{G}^{s,\infty}(\Omega) \cap E'_{3s-1}(\Omega) = D^s(\Omega)$$

Proof. Let $T \in \mathcal{G}^{s,\infty}(\Omega) \cap E'_{3s-1}(\Omega)$, with $\text{supp} T = K$ and ϕ_ε be a net of mollifiers with $\check{\phi} = \phi$ and let $\chi \in D^s(\Omega)$ such that $\chi = 1$ on K . As $[T] \in \mathcal{G}^{s,\infty}(\Omega)$, then $\exists c_1 > 0, \exists k_1 > 0, \exists k_2 > 0, \exists \varepsilon_1 > 0, \forall \varepsilon \leq \varepsilon_1$,

$$\left| \chi \left(\widehat{T * \phi_\varepsilon} \right) (\xi) \right| \leq c_1 e^{k_1 \varepsilon^{-\frac{1}{2s-1}} - k_2 |\xi|^{\frac{1}{s}}},$$

then

$$\begin{aligned} \left| \chi(\widehat{T * \phi_\varepsilon})(\xi) - \widehat{T}(\xi) \right| &= \left| \chi(\widehat{T * \phi_\varepsilon})(\xi) - \chi \widehat{T}(\xi) \right| \\ &= \left| \langle T, (\chi e^{-i\xi \cdot}) * \phi_\varepsilon - (\chi e^{-i\xi \cdot}) \rangle \right| \end{aligned}$$

As $E'_{3s-1}(\Omega) \subset E'_s(\Omega)$, then $\exists L$ a compact subset of Ω such that $\forall h > 0, \exists c > 0$, and

$$\left| \chi(\widehat{T * \phi_\varepsilon})(\xi) - \widehat{T}(\xi) \right| \leq c \sup_{\alpha \in \mathbb{Z}_+^m, x \in L} \frac{h^{|\alpha|}}{\alpha!^s} |(\partial^\alpha (\chi e^{-i\xi \cdot} * \phi_\varepsilon - \chi e^{-i\xi \cdot})(x))|$$

We have $e^{-i\xi} \chi \in D^s(\Omega)$, from the corollary 3-9, $\exists c_2 > 0, \forall k_3 > 0, \exists \eta > 0, \forall \varepsilon \leq \eta$,

$$\sup_{\alpha \in \mathbb{Z}_+^m, x \in L} \frac{c_2^{|\alpha|}}{\alpha!^s} |\partial^\alpha (\chi e^{-i\xi \cdot} * \phi_\varepsilon - \chi e^{-i\xi \cdot})(x)| \leq c_2 e^{-k_3 \varepsilon^{-\frac{1}{2s-1}}},$$

so there exists $c' > 0$, such that

$$\left| \widehat{T}(\xi) - \chi(\widehat{T * \phi_\varepsilon})(\xi) \right| \leq c' e^{-k_3 \varepsilon^{-\frac{1}{2s-1}}}$$

Let $\varepsilon \leq \min(\eta, \varepsilon_1)$, then

$$\begin{aligned} \left| \widehat{T}(\xi) \right| &\leq \left| \widehat{T}(\xi) - \chi(\widehat{T * \phi_\varepsilon})(\xi) \right| + \left| \chi(\widehat{T * \phi_\varepsilon})(\xi) \right| \\ &\leq c' e^{-k_3 \varepsilon^{-\frac{1}{2s-1}}} + c_1 e^{k_1 \varepsilon^{-\frac{1}{2s-1}} - k_2 |\xi|^{\frac{1}{s}}} \end{aligned}$$

Take $c = \max(c', c_1)$, $\varepsilon = \left(\frac{k_1}{(k_2 - r) |\xi|^{\frac{1}{s}}} \right)^{2s-1}$, $r \in]0, k_2[$ and $k_3 = \frac{k_1 r}{k_2 - r}$, then $\exists \delta = 2r > 0, \exists c > 0$ such that

$$\left| \widehat{T}(\xi) \right| \leq c e^{-\delta |\xi|^{\frac{1}{s}}},$$

which means $T \in E^s(\Omega)$. As $\text{supp} T = K$ is a compact then we have $T \in D^s(\Omega)$ \square

5. GENERALIZED GEVREY WAVE FRONT

The defined local regularity and its Fourier characterization, studied in the previous section, allow us to define the fundamental concept of every microlocal analysis, i. e. the generalized Gevrey wave front of a generalized Gevrey ultradistribution.

Definition 10. We define $\sum_g^s(f) \subset \mathbb{R}^m \setminus \{0\}$, $f \in \mathcal{G}_c^s(\Omega)$, as the complement of the set of points having a conic neighborhood Γ such that $\exists k_1 > 0, \exists k_2 > 0, \exists C > 0, \exists \varepsilon_0 \in]0, 1[, \forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$,

$$(33) \quad \left| \widehat{f_\varepsilon}(\xi) \right| \leq c \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - k_2 |\xi|^{\frac{1}{s}} \right)$$

Proposition 12. For every $f \in \mathcal{G}_c^s(\Omega)$, we have

- 1) The set $\sum_g^s(f)$ is a close cone.
- 2) $\sum_g^s(f) = \emptyset \iff f \in \mathcal{G}^{s,\infty}(\Omega)$.
- 3) $\sum_g^s(\psi f) \subset \sum_g^s(f), \forall \psi \in E^s(\Omega)$.

Proof. The proofs of 1) is easy, 2) holds from the proposition 10. Let us proof 3), if $\xi_0 \notin \sum_g^s(f)$, then $\exists \Gamma$ a conic neighborhood of $\xi_0, \exists k_1 > 0, \exists k_2 > 0, \exists c_1 > 0, \exists \varepsilon_0 \in]0, 1[, \forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$,

$$(34) \quad \left| \widehat{f_\varepsilon}(\xi) \right| \leq c_1 \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - k_2 |\xi|^{\frac{1}{s}} \right)$$

Let $\chi \in D^s(\Omega)$, $\chi = 1$ on neighborhood of $\text{supp} f$, so $\chi \psi \in D^s(\Omega)$, hence $\exists k_3 > 0, \exists c_2 > 0, \forall \xi \in \mathbb{R}^m$,

$$(35) \quad \left| \widehat{\chi \psi}(\xi) \right| \leq c_2 \exp \left(-k_3 |\xi|^{\frac{1}{s}} \right),$$

Let Λ be a conic neighborhood of ξ_0 such that, $\bar{\Lambda} \subset \Gamma$, we have, for a fixed $\xi \in \Lambda$,

$$\widehat{\chi\psi f_\varepsilon}(\xi) = \int_A \widehat{f_\varepsilon}(\eta) \widehat{\chi\psi}(\eta - \xi) d\eta + \int_B \widehat{f_\varepsilon}(\eta) \widehat{\chi\psi}(\eta - \xi) d\eta,$$

where $A = \left\{ \eta; |\xi - \eta|^{\frac{1}{s}} \leq \delta \left(|\xi|^{\frac{1}{s}} + |\eta|^{\frac{1}{s}} \right) \right\}$; $B = \left\{ \eta; |\xi - \eta|^{\frac{1}{s}} > \delta \left(|\xi|^{\frac{1}{s}} + |\eta|^{\frac{1}{s}} \right) \right\}$. We choose δ sufficiently small such that $A \subset \Gamma$ and $\frac{|\xi|}{2^s} < |\eta| < 2^s |\xi|$. Then $\exists \varepsilon_0 \in]0, 1[$, $\forall \varepsilon \leq \varepsilon_0$,

$$\begin{aligned} \left| \int_A \widehat{f_\varepsilon}(\eta) \widehat{\chi\psi}(\eta - \xi) d\eta \right| &\leq c_1 c_2 \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - \frac{k_2}{2} |\xi|^{\frac{1}{s}} \right) \times \\ &\quad \times \left| \int_A \exp \left(-k_3 |\eta - \xi|^{\frac{1}{s}} \right) d\eta \right| \\ (36) \quad &\leq c' \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - \frac{k_2}{2} |\xi|^{\frac{1}{s}} \right) \end{aligned}$$

As $f \in \mathcal{G}_c^s(\Omega)$, from proposition 8, $\exists c > 0$, $\exists \mu_1 > 0$, $\exists \varepsilon_1 \in]0, 1[$, $\forall \mu_2 > 0$, $\forall \xi \in \mathbb{R}^m$, $\forall \varepsilon \leq \varepsilon_1$, such that

$$\left| \widehat{f_\varepsilon}(\xi) \right| \leq c \exp \left(\mu_1 \varepsilon^{-\frac{1}{2s-1}} + \mu_2 |\xi|^{\frac{1}{s}} \right),$$

hence, for $\varepsilon \leq \min(\varepsilon_0, \varepsilon_1)$, we have

$$\begin{aligned} \left| \int_B \widehat{f_\varepsilon}(\eta) \widehat{\psi}(\eta - \xi) d\eta \right| &\leq c c_2 \exp \left(\mu_1 \varepsilon^{-\frac{1}{2s-1}} \right) \int_B \exp \left(\mu_2 |\eta|^{\frac{1}{s}} - k_3 |\eta - \xi|^{\frac{1}{s}} \right) d\eta \\ &\leq c'' \exp \left(\mu_1 \varepsilon^{-\frac{1}{2s-1}} \right) \int_B \exp \left(\mu_2 |\eta|^{\frac{1}{s}} - k_3 \delta \left(|\xi|^{\frac{1}{s}} + |\eta|^{\frac{1}{s}} \right) \right) d\eta, \end{aligned}$$

then, taking $\mu_2 < k_3 \delta$, we obtain

$$(37) \quad \left| \int_B \widehat{f_\varepsilon}(\eta) \widehat{\psi}(\eta - \xi) d\eta \right| \leq c \exp \left(\mu_1 \varepsilon^{-\frac{1}{2s-1}} - k_3 \delta |\xi|^{\frac{1}{s}} \right)$$

Consequently, (36) and (37) give $\xi_0 \notin \sum_g^s(\psi f)$ □

Definition 11. Let $f \in \mathcal{G}^s(\Omega)$ and $x_0 \in \Omega$, the cone of s -singular directions of f at x_0 , denoted $\sum_{g, x_0}^s(f)$, is

$$(38) \quad \sum_{g, x_0}^s(f) = \bigcap \left\{ \sum_g^s(\phi f) : \phi \in D^s(\Omega) \text{ and } \phi = 1 \text{ on a neighborhood of } x_0 \right\}$$

Lemma 13. Let $f \in \mathcal{G}^s(\Omega)$, then

$$\sum_{g, x_0}^s(f) = \emptyset \iff x_0 \notin s\text{-singsupp}_g(f)$$

Proof. Let $x_0 \notin s\text{-singsupp}_g(f)$, i.e. $\exists U \subset \Omega$ an open neighborhood of x_0 such that $f \in \mathcal{G}^{s, \infty}(U)$, let $\phi \in D^s(U)$ such that $\phi = 1$ on a neighborhood of x_0 , then $\phi f \in \mathcal{G}^{s, \infty}(\Omega)$. Hence, from the proposition 12, $\sum_g^s(\phi f) = \emptyset$, i.e. $\sum_{g, x_0}^s(f) = \emptyset$.

Suppose now $\sum_{g, x_0}^s(f) = \emptyset$, let $r > 0$ such that $B(0, 2r) \subset \Omega$ and let $\psi \in D^s(B(0, 2r))$ such that $0 \leq \psi \leq 1$ and $\psi = 1$ on $B(0, r)$. Let $\psi_j(x) = \psi(3^j(x - x_0))$ then it is clear that $\text{supp}(\psi_j) \subset B(x_0, \frac{2r}{3^j}) \subset \Omega$ and $\psi_j = 1$ on $B(x_0, \frac{r}{3^j})$, we have $\forall \phi \in D^s(\Omega)$ with $\phi = 1$ on a neighborhood U of x_0 , $\exists j \in \mathbb{Z}^+$ such that $\text{supp}(\psi_j) \subset U$, then $\psi_j f_\varepsilon = \psi_j \phi f_\varepsilon$ and from proposition 12, we have

$$\sum_g^s(\psi_j f) \subset \sum_g^s(\phi f),$$

which gives

$$(39) \quad \bigcap_{j \in \mathbb{Z}^+} \left(\sum_g^s(\psi_j f) \right) = \sum_{g, x_0}^s(f) = \emptyset$$

We have $\psi_j = 1$ on $\text{supp}(\psi_{j+1})$, then $\sum_g^s(\psi_{j+1}f) \subset \sum_g^s(\psi_j f)$, so from (39), there exists $n \in \mathbb{Z}^+$ sufficiently large such that $(\psi_n f) \in \mathcal{G}^{s,\infty}(\Omega)$, then $f \in \mathcal{G}^{s,\infty}(B(x_0, \frac{r}{3^n}))$, which means. $x_0 \notin s\text{-singsupp}_g(f)$ \square

Now, we are ready to give the definition of the generalized Gevrey wave front and its main properties.

Definition 12. A point $(x_0, \xi_0) \notin WF_g^s(f) \subset \Omega \times \mathbb{R}^m \setminus \{0\}$ if $\xi_0 \notin \sum_{g,x_0}^s(f)$, i.e. there exists $\phi \in D^s(\Omega)$, $\phi(x) = 1$ neighborhood of x_0 , and conic neighborhood Γ of ξ_0 , $\exists k_1 > 0, \exists k_2 > 0, \exists c > 0, \exists \varepsilon_0 \in]0, 1[$, such that $\forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$,

$$\left| \widehat{\phi f_\varepsilon}(\xi) \right| \leq c \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - k_2 |\xi|^{\frac{1}{s}} \right)$$

The main properties of the generalized Gevrey wave front WF_g^s are given in the following proposition.

Proposition 14. Let $f \in \mathcal{G}^s(\Omega)$, then

- 1) The projection of $WF_g^s(f)$ on Ω is the s - $singsupp_g(f)$
- 2) If $f \in \mathcal{G}_c^s(\Omega)$, then the projection of $WF_g^s(f)$ on $\mathbb{R}^m \setminus \{0\}$ is $\sum_g^s(f)$
- 3) $\forall \alpha \in \mathbb{Z}_+^m, WF_g^s(\partial^\alpha f) \subset WF_g^s(f)$
- 4) $\forall g \in \mathcal{G}^{s,\infty}(\Omega), WF_g^s(gf) \subset WF_g^s(f)$

Proof. 1) and 2) holds from the definition, the proposition 12 and lemma 13. 3) Let $(x_0, \xi_0) \notin WF_g^s(f)$, then $\exists \phi \in D^s(\Omega)$, $\phi(x_0) = 1$ on a neighborhood \bar{U} of x_0 , there exist a conic neighborhood Γ of ξ_0 , $k_1 > 0, k_2 > 0, c_1 > 0, \varepsilon_0 \in]0, 1[$, such that $\forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$,

$$(40) \quad \left| \widehat{\phi f_\varepsilon}(\xi) \right| \leq c_1 \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - k_2 |\xi|^{\frac{1}{s}} \right)$$

We have, for $\psi \in D^s(U)$ such that $\psi(x_0) = 1$,

$$\begin{aligned} \left| \widehat{\psi \partial f_\varepsilon}(\xi) \right| &= \left| \widehat{\partial(\psi f_\varepsilon)}(\xi) - \widehat{(\partial \psi) f_\varepsilon}(\xi) \right| \\ &\leq |\xi| \left| \widehat{\psi \phi f_\varepsilon}(\xi) \right| + \left| \widehat{(\partial \psi) \phi f_\varepsilon}(\xi) \right| \end{aligned}$$

As $WF_g^s(\psi f) \subset WF_g^s(f)$, then (40) holds for both $\left| \widehat{\psi \phi f_\varepsilon}(\xi) \right|$ and $\left| \widehat{(\partial \psi) \phi f_\varepsilon}(\xi) \right|$. So

$$\begin{aligned} |\xi| \left| \widehat{\psi \phi f_\varepsilon}(\xi) \right| &\leq c |\xi| \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - k_2 |\xi|^{\frac{1}{s}} \right) \\ &\leq c' \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - k_3 |\xi|^{\frac{1}{s}} \right), \end{aligned}$$

with $c' > 0, k_3 > 0$ such that $|\xi| \leq c' \exp(k_2 - k_3) |\xi|^{\frac{1}{s}}$. Hence (40) holds for $\left| \widehat{\psi \partial f_\varepsilon}(\xi) \right|$, which proves $(x_0, \xi_0) \notin WF_g^s(\partial f)$

4) Let $(x_0, \xi_0) \notin WF_g^s(f)$, then $\exists \phi \in D^s(\Omega)$, $\phi(x) = 1$ on a neighborhood U of x_0 , there exist a conic neighborhood Γ of ξ_0 , $k_1 > 0, k_2 > 0, c_1 > 0, \varepsilon_0 \in]0, 1[$, such that $\forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_0$,

$$\left| \widehat{\phi f_\varepsilon}(\xi) \right| \leq c_1 \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - k_2 |\xi|^{\frac{1}{s}} \right)$$

Let $\psi \in D^s(\Omega)$ and $\psi = 1$ on $\text{supp} \phi$, then $\widehat{\phi g_\varepsilon f_\varepsilon} = \widehat{\psi g_\varepsilon} * \widehat{\phi f_\varepsilon}$. We have $\psi g \in \mathcal{G}^{s,\infty}(\Omega)$, then $\exists c_2 > 0, \exists k_3 > 0, \exists k_4 > 0, \exists \varepsilon_1 > 0, \forall \xi \in \mathbb{R}^m, \forall \varepsilon \leq \varepsilon_1$,

$$\left| \widehat{\psi g_\varepsilon}(\xi) \right| \leq c_2 \exp \left(k_3 \varepsilon^{-\frac{1}{2s-1}} - k_4 |\xi|^{\frac{1}{s}} \right),$$

so

$$\widehat{\phi g_\varepsilon f_\varepsilon}(\xi) = \int_A \widehat{\phi f_\varepsilon}(\eta) \widehat{\psi g_\varepsilon}(\eta - \xi) d\eta + \int_B \widehat{\phi f_\varepsilon}(\eta) \widehat{\psi g_\varepsilon}(\eta - \xi) d\eta,$$

where A and B are the same as in the proof of proposition 12. By proposition 8, we have $\exists c > 0, \exists \mu_1 > 0, \forall \mu_2 > 0, \exists \varepsilon_2 > 0, \forall \xi \in \mathbb{R}^m, \forall \varepsilon \leq \varepsilon_2$,

$$\left| \widehat{\phi f_\varepsilon}(\xi) \right| \leq c \exp \left(\mu_1 \varepsilon^{-\frac{1}{2s-1}} + \mu_2 |\xi|^{\frac{1}{s}} \right)$$

The same steps as the proposition 12 finish the proof \square

Corollary 15. *Let $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be a partial differential operator with $\mathcal{G}^{s, \infty}(\Omega)$ coefficient, then*

$$WF_g^s(P(x, D)f) \subset WF_g^s(f), \forall f \in \mathcal{G}^s(\Omega)$$

Remark 6. *The reverse inclusion will give a generalized Gevrey microlocal hypoellipticity of linear partial differential operators with generalized coefficients. The case of generalized \mathcal{G}^∞ -microlocal hypoellipticity has been studied recently in [16].*

6. GENERALIZED HÖRMANDER'S THEOREM

We define $WF_g^s(f) + WF_g^s(g)$, where $f, g \in \mathcal{G}^s(\Omega)$, as the set

$$\{(x, \xi + \eta); (x, \xi) \in WF_g^s(f), (x, \eta) \in WF_g^s(g)\}$$

Lemma 16. *Let Σ_1, Σ_2 be closed cones in $\mathbb{R}^m \setminus \{0\}$, such that $0 \notin \Sigma_1 + \Sigma_2$, then*

- i) $\overline{\Sigma_1 + \Sigma_2}^{\mathbb{R}^m \setminus \{0\}} = (\Sigma_1 + \Sigma_2) \cup \Sigma_1 \cup \Sigma_2$
- ii) *For any open conic neighborhood Γ of $\Sigma_1 + \Sigma_2$ in $\mathbb{R}^m \setminus \{0\}$, one can find open conic neighborhoods of Γ_1, Γ_2 in $\mathbb{R}^m \setminus \{0\}$ of, respectively, Σ_1, Σ_2 , such that*

$$\Gamma_1 + \Gamma_2 \subset \Gamma$$

Proof. See [15] \square

Theorem 17. *Let $f, g \in \mathcal{G}^s(\Omega)$, such that $\forall x \in \Omega$,*

$$(41) \quad (x, 0) \notin WF_g^s(f) + WF_g^s(g),$$

then

$$(42) \quad WF_g^s(fg) \subseteq (WF_g^s(f) + WF_g^s(g)) \cup WF_g^s(f) \cup WF_g^s(g)$$

Proof. Let $(x_0, \xi_0) \notin (WF_g^s(f) + WF_g^s(g)) \cup WF_g^s(f) \cup WF_g^s(g)$, then $\exists \phi \in D^s(\Omega)$, $\phi(x_0) = 1$, $\xi_0 \notin \left(\sum_g^s(\phi f) + \sum_g^s(\phi g) \right) \cup \sum_g^s(\phi f) \cup \sum_g^s(\phi g) = \overline{\sum_g^s(\phi f) + \sum_g^s(\phi g)}^{\mathbb{R}^m \setminus \{0\}}$. Let $\bar{\Gamma}_0$ be an open conic neighborhood of $\sum_g^s(\phi f) + \sum_g^s(\phi g)$ in $\mathbb{R}^m \setminus \{0\}$ such that $\xi_0 \notin \bar{\Gamma}_0$. By lemma 16 there exist open cones Γ_1 and Γ_2 in $\mathbb{R}^m \setminus \{0\}$ such that

$$\sum_g^s(\phi f) \subset \Gamma_1, \quad \sum_g^s(\phi g) \subset \Gamma_2 \quad \text{and} \quad \Gamma_1 + \Gamma_2 \subset \bar{\Gamma}_0$$

Define $\Gamma = \mathbb{R}^m \setminus \bar{\Gamma}_0$, so

$$(43) \quad \Gamma \cap \Gamma_2 = \emptyset \quad \text{and} \quad (\Gamma - \Gamma_2) \cap \Gamma_1 = \emptyset$$

Let $\xi \in \Gamma$ and $\varepsilon \in]0, 1[$

$$\begin{aligned} \widehat{\phi f_\varepsilon \phi g_\varepsilon}(\xi) &= \left(\widehat{\phi f_\varepsilon} * \widehat{\phi g_\varepsilon} \right)(\xi) \\ &= \int_{\Gamma_2} \widehat{\phi f_\varepsilon}(\xi - \eta) \widehat{\phi g_\varepsilon}(\eta) d\eta + \int_{\Gamma_2^c} \widehat{\phi f_\varepsilon}(\xi - \eta) \widehat{\phi g_\varepsilon}(\eta) d\eta \\ &= I_1(\xi) + I_2(\xi) \end{aligned}$$

From (43), $\exists C_1 > 0, \exists k_1, k_2 > 0, \exists \varepsilon_1 > 0$ such that $\forall \varepsilon \leq \varepsilon_1, \forall \eta \in \Gamma_2$,

$$\widehat{\phi f_\varepsilon}(\xi - \eta) \leq C_1 \exp \left(k_1 \varepsilon^{-\frac{1}{2s-1}} - k_2 |\xi - \eta|^{\frac{1}{s}} \right),$$

and from proposition 8 $\exists C_2 > 0, \exists k_3 > 0, \forall k_4 > 0, \exists \varepsilon_2 > 0, \forall \eta \in \mathbb{R}^m, \forall \varepsilon \leq \varepsilon_2$,

$$\left| \widehat{\phi g_\varepsilon}(\eta) \right| \leq C_2 \exp \left(k_2 \varepsilon^{-\frac{1}{2s-1}} + k_4 |\eta|^{\frac{1}{s}} \right)$$

Let $\gamma > 0$ sufficiently small such that $|\xi - \eta|^{\frac{1}{s}} \geq \gamma \left(|\xi|^{\frac{1}{s}} + |\eta|^{\frac{1}{s}} \right), \forall \eta \in \Gamma_2$. Hence for $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$,

$$|I_1(\xi)| \leq C_1 C_2 \exp \left((k_1 + k_2) \varepsilon^{-\frac{1}{2s-1}} - k_2 \gamma |\xi|^{\frac{1}{s}} \right) \int \exp \left(-k_2 \gamma |\eta|^{\frac{1}{s}} + k_4 |\eta|^{\frac{1}{s}} \right) d\eta$$

take $k_4 > k_2 \gamma$, then

$$(44) \quad |I_1(\xi)| \leq C' \exp \left(k'_1 \varepsilon^{-\frac{1}{2s-1}} - k'_2 |\xi|^{\frac{1}{s}} \right)$$

Let $r > 0$,

$$\begin{aligned} I_2(\xi) &= \int_{\Gamma_2^c \cap \{|\eta| \leq r|\xi|\}} \widehat{\phi f_\varepsilon}(\xi - \eta) \widehat{\phi g_\varepsilon}(\eta) d\eta + \int_{\Gamma_2^c \cap \{|\eta| \geq r|\xi|\}} \widehat{\phi f_\varepsilon}(\xi - \eta) \widehat{\phi g_\varepsilon}(\eta) d\eta \\ &= I_{21}(\xi) + I_{22}(\xi) \end{aligned}$$

Choose r sufficiently small such that $\left\{ |\eta|^{\frac{1}{s}} \leq r |\xi|^{\frac{1}{s}} \right\} \implies \xi - \eta \notin \Gamma_1$. Then $|\xi - \eta|^{\frac{1}{s}} \geq (1 - r) |\xi|^{\frac{1}{s}} \geq (1 - 2r) |\xi|^{\frac{1}{s}} + |\eta|^{\frac{1}{s}}$, consequently $\exists C_3 > 0, \exists \lambda_1, \lambda_2, \lambda_3 > 0, \exists \varepsilon_3 > 0$ such that $\forall \varepsilon \leq \varepsilon_3$,

$$\begin{aligned} |I_{21}(\xi)| &\leq C_3 \exp \left(\lambda_1 \varepsilon^{-\frac{1}{2s-1}} \right) \int \exp \left(-\lambda_2 |\xi - \eta|^{\frac{1}{s}} - \lambda_3 |\eta|^{\frac{1}{s}} \right) d\eta \\ &\leq C_3 \exp \left(\lambda_1 \varepsilon^{-\frac{1}{2s-1}} - \lambda'_2 |\xi|^{\frac{1}{s}} \right) \int \exp \left(-\lambda'_3 |\eta|^{\frac{1}{s}} \right) d\eta \\ &\leq C'_3 \exp \left(\lambda_1 \varepsilon^{-\frac{1}{2s-1}} - \lambda'_2 |\xi|^{\frac{1}{s}} \right) \end{aligned}$$

If $|\eta|^{\frac{1}{s}} \geq r |\xi|^{\frac{1}{s}}$, we have $|\eta|^{\frac{1}{s}} \geq \frac{|\eta|^{\frac{1}{s}} + r |\xi|^{\frac{1}{s}}}{2}$, and then $\exists C_4 > 0, \exists \mu_1, \mu_3 > 0, \forall \mu_2 > 0, \exists \varepsilon_4 > 0$ such that $\forall \varepsilon \leq \varepsilon_4$,

$$\begin{aligned} |I_{21}(\xi)| &\leq C_4 \exp \left(\mu_1 \varepsilon^{-\frac{1}{2s-1}} \right) \int \exp \left(\mu_2 |\xi - \eta|^{\frac{1}{s}} - \mu_3 |\eta|^{\frac{1}{s}} \right) d\eta \\ &\leq C_4 \exp \left(\mu_1 \varepsilon^{-\frac{1}{2s-1}} \right) \int \exp \left(\mu_2 |\xi - \eta|^{\frac{1}{s}} - \mu'_3 |\eta|^{\frac{1}{s}} - \mu'_3 |\xi|^{\frac{1}{s}} \right) d\eta, \end{aligned}$$

if take $\mu_2 < \frac{\mu'_3}{2} \left(1 + \frac{1}{r} \right)$, we obtain

$$|I_{21}(\xi)| \leq C'_4 \exp \left(k'_3 \varepsilon^{-\frac{1}{2s-1}} - \mu'_3 |\xi|^{\frac{1}{s}} \right),$$

which finish the proof □

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